Contents

Preface

This book grew out of a lecture course taught at the University of Melbourne over a series of years. The audience was third-year actuarial students who partially gained an exemption from the Institute of Actuaries CT8 module if they did well. The nature of the audience and the exemption placed certain constraints on the syllabus and delivery that made it hard to find a suitable textbook. The graduate level texts simply being too hard, whilst the undergraduate and MBA books did not cover the mathematics in sufficient depth. In particular, the students were fairly mathematical but more oriented towards computations than proof. In addition, the choice of topics had to be tuned to the actuarial syllabus and that is reflected in this book.

In terms of mathematical level, we strive to achieve a mid-level where mathematics is not shied away from nor hived off to appendices, but also not so hard as to deter the undergraduate reader. Also, this being a book on mathematical portfolio theory, the mathematics takes centre stage for most of the book: our objective is to study the mathematics of portfolio theory without losing sight of the finance. As both authors have been both practitioners and academics, a theme throughout is that a model is a model and not reality, and we aim to highlight our assumptions and their consequences. We provide a lot of problems with solutions in the belief that this is ultimately how the material is best learnt, and as a consequence of the fact that students always want more problems and more solutions.

We first look at the definitions of risk and return. We then explore Markowitz's portfolio theory. We start with the two-asset case, then add a riskless asset, and finally treat the general case. We derive a couple of different ways to find efficient portfolios in that case. We then move on to seeing how simplifying

correlation structures can help to reduce the amount of data needed to estimate the model parameters.

We then make a long excursion into utility theory, looking at both its pros and cons. We also look at the offshoots of stochastic dominance and geometric mean maximization. An important issue in any insurance company or bank is risk control, and we therefore look at risk measures including VAR and conditional expected shortfall. We also examine the coherence axioms.

We then move onto a critical look at the capital asset pricing model and the arbitrage pricing theory. We also discuss market efficiency and rationality. Here we adopt a more discursive viewpoint. We finish by looking at long-term models of stock prices using Brownian motion and the Wilkie model.

Much of this book has been shaped by interactions with our students and from their explicit feedback, and we thank them all for their input.

For the reader whose appetite has been whetted and wishes to study the material in greater depth, we mention a few books which we have found helpful.

- Elton *et al*, [6], is very discursive and contains very detailed references but is much less mathematical than this book.
- Pennachi, [14], is a nice text aimed at PhD students.
- Cochrane, [3], has both good discussion and good mathematics and will reward the reader who perseveres.
- Markowitz's original book, *Portfolio Selection,* [11], is still a good read.

Mark Joshi
Jane Paterson
Melbourne 2012

1
Definitions of risk and return

1.1 Introduction

Mathematics can be applied to the practice of finance in a number of ways. These include trying to use mathematics to predict asset price movements (*statistical arbitrage*), measuring and controlling risk in trading books (*risk management,*) pricing options and other contingent claims by assessing hedging strategies (*derivatives pricing,*) and the use of mathematics to maximize the risk-return trade-off when investing in the markets: *portfolio theory*. It is portfolio theory which we will address in this volume. This subject is sometimes called *modern portfolio theory* or *MPT*.

At first glance the objective of maximizing the risk-return trade-off when investing in the markets appears straightforward and intuitive; that is, any rational investor will want to maximize the anticipated return on his or her investment whilst minimizing the risk of unexpected loss. However, in order to apply the rigour of mathematics to this activity, we need first to carefully define these terms. What is risk? What is return? How do we decide the trade-off between them? There are multiple ways to do this, and we will examine the more widely-used ones.

Throughout we will make two fundamental assumptions. The first is that individual assets are correctly priced. This means that "stock picking" is pointless and, accordingly, our efforts will focus on how to compare portfolios with each other.

Secondly, to ensure consistency we will generally work across a fixed timeframe, for example, one year. We should think of ourselves as a funds manager whose performance is assessed on a yearly basis. The funds manager will be